PHYS 101 – General Physics I Midterm Exam

Duration: 120 minutes | Saturday, 18 November 2023; 09:30

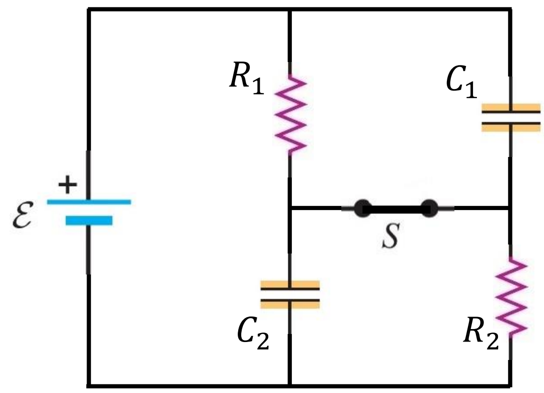
1. Switch S shown in the figure has been closed for a very long time, and a constant current has been established in the circuit.

(a) (5 Pts.) Find the charge on the capacitors C_1 and C_2.

(b) (5 Pts.) How much power is supplied by the battery?

(c) (10 Pts.) Now the switch S is opened. Find the current through the resistor R_1 as a function of time.

(d) (5 Pts.) What will be final charges on the capacitors after a very long time?



Solution:

(a) In this case both capacitors are fully charged, and hence, no current passes through them. The current through the equivalent circuit is $I = \mathcal{E}/(R_1 + R_2)$. Voltage over C_1 is equal to the voltage over R_1, which is $V_1 = R_1 I = \mathcal{E}R_1/(R_1 + R_2)$, while voltage over C_2 is equal to the voltage over R_2, which is $V_2 = R_2 I = \mathcal{E}R_2/(R_1 + R_2)$. Hence, final charges on the capacitors are

$$Q_1 = C_1 V_1 = \mathcal{E}R_1 C_1/(R_1 + R_2), \qquad Q_2 = \mathcal{E}R_2 C_2/(R_1 + R_2)\,.$$

(b)

$$P = \mathcal{E}I = \frac{\mathcal{E}^2}{R_1 + R_2}.$$

Note that this is equal to the sum of powers dissipated by the two resistors since $P_R = R_1 I^2 + R_2 I^2 = P$

(c) When the switch is opened the loop equation with R_1 is written as

$$\mathcal{E} - R_1 i_1 - V_{C2} = 0 \;\rightarrow\; \mathcal{E} - R_1 \frac{dq_2}{dt} - \frac{q_2}{C_2} = 0 \;\rightarrow\; \frac{dq'_2}{q'_2 - \mathcal{E}C_2} = -\frac{dt'}{R_1 C_2}.$$

Integrating the left hand side from the initial charge Q_{20} to q_2,and integrating the right hand side from $t' = 0$ to t, we get

$$q_2(t) = \mathcal{E}C_2 + (Q_{20} - \mathcal{E}C_2)e^{-t/R_1C_2} \rightarrow \quad q_2(t) = \mathcal{E}C_2 + \left(\frac{\mathcal{E}R_2C_2}{R_1 + R_2} - \mathcal{E}C_2\right)e^{-t/R_1C_2} = \mathcal{E}C_2\left(1 - \frac{R_1 e^{-t/R_1C_2}}{R_1 + R_2}\right)$$

$$i_1 = \frac{dq_2}{dt} = \left(\frac{\mathcal{E}}{R_1 + R_2}\right)e^{-t/R_1C_2}.$$

(d) With the switch open and both capacitors fully charged no current will flow in the circuit. Therefore

$$\lim_{t\to\infty} q_1(t) = \mathcal{E}C_1\,, \qquad \lim_{t\to\infty} q_2(t) = \mathcal{E}C_2\,.$$

2. (25 Pts.) A particle with charge q ($q > 0$) and mass m, initially (at time $t = 0$) moving with velocity $\vec{\mathbf{v}}_0 = v_0(\frac{1}{2}\,\hat{\mathbf{i}} + \frac{\sqrt{3}}{2}\,\hat{\mathbf{j}})$ at the origin, enters the region $x > 0,\ y > 0$ where a uniform magnetic field $\vec{\mathbf{B}} = B_0\,\hat{\mathbf{k}}$ exists. At a later time t_α, the particle will cross the x-axis at $x = \alpha$.

(a) (4 Pts.) Sketch the motion of the particle on the xy-plane.

(b) (7 Pts.) Determine α in terms of q, m, v_0, and B_0.

(c) (7 Pts.) Determine t_α in terms of q, m, v_0, and B_0.

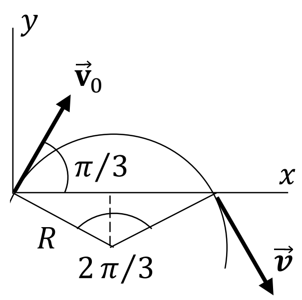
(d) (7 Pts.) Find the velocity of the particle at the instant $t = t_\alpha$.

Solution: Force on the charged particle at the instant it enters the region is

$$\vec{\mathbf{F}} = q\,\vec{\mathbf{v}} \times \vec{\mathbf{B}} = qv_0\left(\frac{1}{2}\,\hat{\mathbf{i}} + \frac{\sqrt{3}}{2}\,\hat{\mathbf{j}}\right) \times B_0\,\hat{\mathbf{k}} \;\rightarrow\; \vec{\mathbf{F}} = qv_0B_0\left(\frac{\sqrt{3}}{2}\,\hat{\mathbf{i}} - \frac{1}{2}\,\hat{\mathbf{j}}\right).$$

During the motion of the particle in the region the force will always be perpendicular to its velocity with magnitude $F = qv_0B_0$. Therefore, the path of the particle will be a circle with radius R, such that

$$F = \frac{mv_0^2}{R} \;\rightarrow\; qv_0B_0 = \frac{mv_0^2}{R} \;\rightarrow\; R = \frac{mv_0}{qB_0}.$$



From the figure, we see that

$$\alpha = 2R\cos(\pi/6) \;\rightarrow\; \alpha = \sqrt{3}\,\frac{mv_0}{qB_0}.$$

Period of the motion around the circle is

$$T = 2\pi R/v_0 \;\rightarrow\; T = \frac{2\pi m}{qB_0}.$$

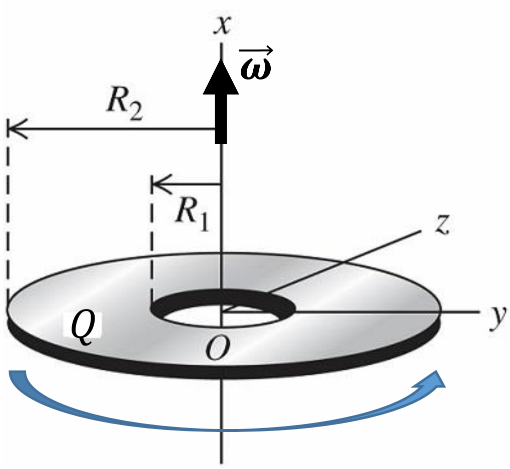
The portion of the arc in the region $y > 0$ is one third of the total circumference. Therefore,

$$t_\alpha = \frac{T}{3} \;\rightarrow\; t_\alpha = \frac{2\pi m}{3qB_0}.$$

From the symmetry of the figure we write

$$\vec{\mathbf{v}}(t_\alpha) = v_0\left(\frac{1}{2}\,\hat{\mathbf{i}} - \frac{\sqrt{3}}{2}\,\hat{\mathbf{j}}\right).$$

3 . A thin disk with a circular hole at its center, called an *annulus*, has inner radius R_1 and outer radius R_2. The disk has a positive charge Q distributed uniformly on its surface. The disk is set spinning with angular velocity ω about an axis perpendicular to the plane of the disk through its center, as shown in the figure.



(a) (10 Pts.) Determine its magnetic dipole moment vector.

(b) (10 Pts.) Determine the magnetic field at the origin.

(c) (5 Pts.) What will be the torque vector on the rotating annulus when it is placed in an external uniform magnetic field $\vec{\mathbf{B}}_{\text{ex}} = B_x\,\hat{\mathbf{i}} + B_z\,\hat{\mathbf{k}}$.

Solution: (a) Divide the annulus into infinitesimal rings with radius r and thickness dr. For each ring, we have

$$d\mu = dI\,A\,, \qquad dI = f dq = \left(\frac{\omega}{2\pi}\right)\sigma(2\pi r dr)\,, \qquad \sigma = \frac{Q}{\pi(R_2^2 - R_1^2)}\,, \qquad A = \pi r^2\,.$$

Therefore

$$d\mu = \left(\frac{Q\omega}{R_2^2 - R_1^2}\right)r^3 dr \quad \rightarrow \quad \mu = \frac{Q\omega}{R_2^2 - R_1^2}\int_{R_1}^{R_2} r^3 dr \quad \rightarrow \quad \mu = \left(\frac{Q\omega}{R_2^2 - R_1^2}\right)\frac{1}{4}(R_2^4 - R_1^4) = \frac{Q\omega}{4}(R_1^2 + R_2^2)\,.$$

$$\vec{\boldsymbol{\mu}} = \frac{Q\omega}{4}(R_1^2 + R_2^2)\,\hat{\mathbf{i}}\,.$$

(b) Magnetic field at the center of a current ring of radius r and infinitesimal thickness dr is directed along the symmetry axis with magnitude

$$dB_x = \frac{\mu_0}{2r}dI\,, \qquad dI = \left(\frac{Q\omega}{\pi(R_2^2 - R_1^2)}\right)r dr \quad \rightarrow \quad dB_x = \frac{\mu_0 Q\omega}{2\pi(R_2^2 - R_1^2)}\,dr\,.$$

Integrating over the annulus, we get

$$B_x = \int_{R_1}^{R_2} dB_x = \frac{\mu_0 Q\omega}{2\pi(R_2^2 - R_1^2)}\int_{R_1}^{R_2} dr = \frac{\mu_0 Q\omega}{2\pi(R_2^2 - R_1^2)}(R_2 - R_1) = \frac{\mu_0 Q\omega}{2\pi(R_2 + R_1)}\,.$$

$$\vec{\mathbf{B}} = \frac{\mu_0 Q\omega}{2\pi(R_2 + R_1)}\,\hat{\mathbf{i}}\,.$$

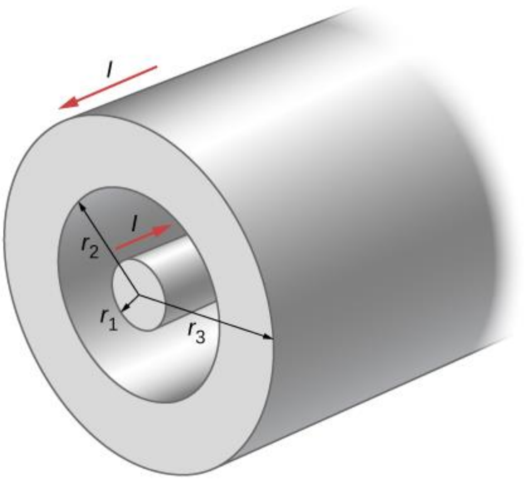
(c)

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}}_{\text{ex}} \quad \rightarrow \quad \vec{\boldsymbol{\tau}} = \frac{Q\omega}{4}(R_1^2 + R_2^2)\,\hat{\mathbf{i}} \times \left(B_x\,\hat{\mathbf{i}} + B_z\,\hat{\mathbf{k}}\right) = -\frac{Q\omega}{4}(R_1^2 + R_2^2)B_z\,\hat{\mathbf{j}}\,.$$

4. A portion of a long, cylindrical coaxial cable is shown in the accompanying figure. Current I flows in the center conductor, and this current is returned in the outer conductor. Assume that the current is distributed uniformly over the cross sections of the two parts of the cable.

(a) (5 Pts.) Find the magnitude of the current density $|\vec{\mathbf{J}}|$ in the regions $r < r_1$ and $r_2 < r < r_3$.

(b) (20 Pts.) Determine the magnitude of the magnetic field in the regions $r < r_1$, $r_1 < r < r_2$, $r_2 < r < r_3$ and $r_3 < r$.



Solution: (a)

$$J = \frac{I}{\pi r_1^2}, \qquad r < r_1, \qquad J = \frac{I}{\pi(r_3^2 - r_2^2)}, \qquad r_2 < r < r_3 .$$

(b) The problem has cylindrical symmetry. We use Ampère's law where the closed path is a circle with radius r centered at the symmetry axis.

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{r}} = \mu_0 I_{\text{enc}} \;\rightarrow\; 2\pi r\, B(r) = \mu_0 (J A_{\text{enc}}) \rightarrow \; B(r) = \frac{\mu_0 J A_{\text{enc}}}{2\pi r}$$

where A_{enc} is the area enclosed carrying current. For $r < r_1$ we have $A_{\text{enc}} = \pi r^2$. Therefore

$$B(r) = \frac{\mu_0}{2\pi r}\left(\frac{I}{\pi r_1^2}\right)(\pi r^2) \;\rightarrow\; B(r) = \frac{\mu_0 I r}{2\pi r_1^2}, \qquad r < r_1 .$$

In the region $r_1 < r < r_2$ we have $I_{\text{enc}} = I$ because $A_{\text{enc}} = \pi r_1^2$. Therefore

$$B(r) = \frac{\mu_0 I}{2\pi r}, \qquad r_1 < r < r_2 .$$

In the region $r_2 < r < r_3$ we have a current I flowing in one direction and part of a current I flowing in the opposite direction where

$$A_{\text{enc}} = \pi(r^2 - r_2^2).$$

Therefore

$$I_{\text{enc}} = I - J A_{\text{enc}} \;\rightarrow\; I_{\text{enc}} = I - \frac{I\pi(r^2 - r_2^2)}{\pi(r_3^2 - r_2^2)} \;\rightarrow\; I_{\text{enc}} = I\left(1 - \frac{r^2 - r_2^2}{r_3^2 - r_2^2}\right).$$

Hence

$$B(r) = \frac{\mu_0 I}{2\pi r}\left(1 - \frac{r^2 - r_2^2}{r_3^2 - r_2^2}\right) = \frac{\mu_0 I}{2\pi r}\left(\frac{r_3^2 - r^2}{r_3^2 - r_2^2}\right), \qquad r_2 < r < r_3 .$$

In the region $r > r_3$ net current enclosed is zero. Therefore

$$B(r) = 0, \qquad r > r_3 .$$